Lyapunov stability in dynamical systems

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April 25, 2014

Dynamical System Framework

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- $\Omega = \mathrm{Dom}(f) \subset \mathbb{R}$
- Phase space $\mathcal{O} \subset \mathbb{R}^n$ contains all of the possible states $\{\vec{x}_i\}$ of a dynamical system.

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- Transform ODE system into integral equations

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- Extend contraction to find maximum interval of existence

Picard Existence Theorem

- Theorem: Let $K \subset \Omega$ be a compact neighborhood of the initial conditions $(t_0, \vec{x}_0)$. If f is a continuous function that is locally Lipschitz, then $\exists \delta > 0$ such that that for every $(t_0, \vec{x}_0)\ K$,

$$\left| f(t, \vec{x}_1) - f(t, \vec{x}_2) \right| \leq L \left| \vec{x}_1 - \vec{x}_2 \right|, \forall \vec{x}_1, \vec{x}_2 \in U$$

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- Continuity of f gives existence of solutions.
- Locally Lipschitz condition of f gives uniqueness of solutions.

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  - $DE(\vec{x})\, f(\vec{x}) \leq 0,\ \forall \vec{x} \in K \setminus \{\vec{x}_{crit}\}$

Lyapunov Stability: Example

- Example:

$$\begin{aligned} \dot{x} &= -2y^3 \\ \dot{y} &= x - 3y^3 \end{aligned}$$