Topological construction in the language of categories

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December 3, 2013

Categorical Framework

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- Composition of arrows is defined when compatible, the source object of one is the target object of the other
- Composition of arrows is associative
- If f is any arrow with source a and target b, then $\mathrm{Id}_b \circ f = f = f \circ \mathrm{Id}_a$

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- Not all categories have initial or terminal objects

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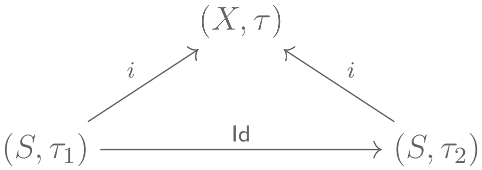

  whenever the identity map $(S, \tau_1) \xrightarrow{\text{Id}} (S, \tau_2)$ is continuous.

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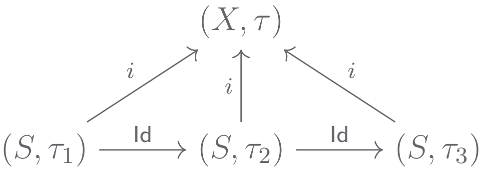

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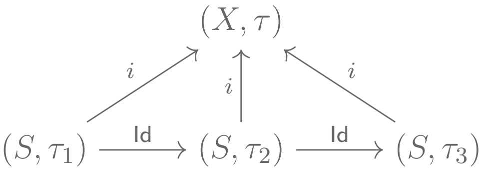

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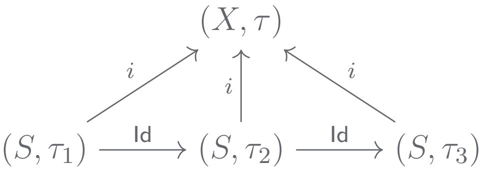

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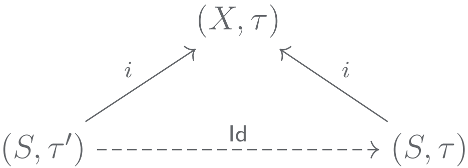

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\begin{array}{ccccc}
X & \xleftarrow{\;f\;} & Z & \xrightarrow{\;g\;} & Y \\
\text{Id}\,\Big\| & & \Big\downarrow{\scriptstyle\varphi} & & \Big\|\,\text{Id} \\
X & \xleftarrow{\;h\;} & W & \xrightarrow{\;k\;} & Y
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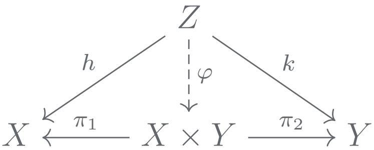

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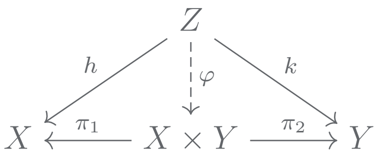

- The definition of $\mathscr{C}_{X,Y}$ can be slightly altered to obtain infinite product of spaces with no effort.

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- Let $\mathscr{C} = \mathbf{Top}$ and let $\Delta : \mathbf{Top} \to \mathbf{Top} \times \mathbf{Top}$ be the diagonal functor with $\Delta X = (X, X)$. Another way to state the previous result is that the pair of projections (π_1, π_2) is a *universal arrow* from the diagonal functor Δ to the object (X, Y).

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- If we let $\mathscr{C} = \textbf{Grp}, \textbf{Ab}, \textbf{Rng}, \textbf{Mod}\text{-}R, \textbf{Set}, \textbf{Cat}, \ldots$ we obtain the product of groups, abelian groups, rings, R-modules, sets, categories, etc as terminal objects in the corresponding category.

Coproducts

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- For the same fixed topological spaces X, Y, let $\mathscr{C}^{\mathsf{op}}_{X,Y}$ be the category defined previously, with arrows reversed. Then the disjoint union $X \sqcup Y$ along with the obvious inclusions $i_1 : X \to X \sqcup Y$, $i_1 : Y \to X \sqcup Y$ is an initial object in $\mathscr{C}^{\mathsf{op}}_{X,Y}$:

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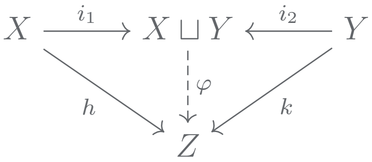

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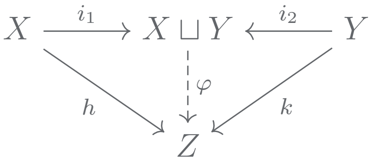

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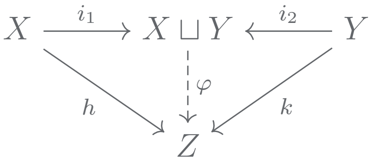

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- All categorical statements about products hold for coproducts by reversing the arrows. This is called *duality*.

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- In **Set**, the coproduct gives disjoint union of sets; in **Top**$_*$ we get the wedge product, in **Ab** or R-**Mod** we get direct sums and in **CRng** we get tensor products.

Quotient spaces

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$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & W \end{array}$$

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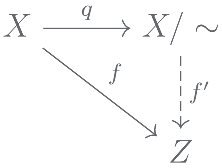

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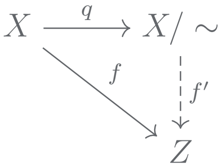

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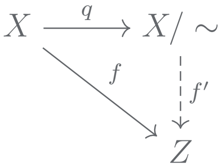

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- The same construction can be used to describe quotients of groups by normal groups, quotients of rings by two-sided ideals, etc. The universality property can even derive the isomorphism theorems in Group theory with no reference to cosets!